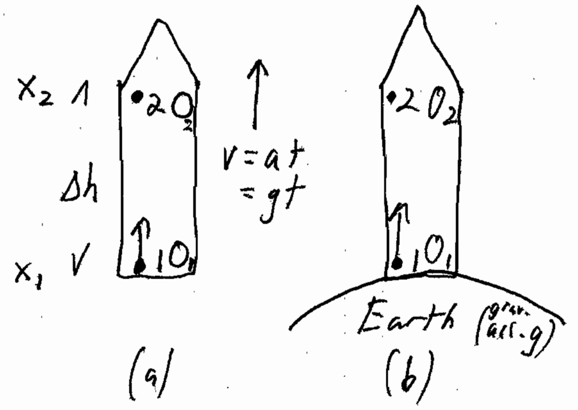
Understanding (?)(*) gravitational redshift & gravitational time dilation

(*) - or get an idea of - - -

① Gravitational redshift as phenomena.



(a) Rocket accelerating. - eg. with
(b) Rocket at rest in Earth grav. field.
Two identical light sources, monochromatic.
Observer O_1 measures freq. f of source 1.
Observer O_2 measures freq. f of source 2.
Light from 1 arrives at 2.
Observer O_2 measures freq. f' of light from 1. $f' < f$! (or $\lambda' > \lambda$).
So light from 1 is red-shifted when observed by O_2.

Remember: speed of light is c always. (in vacuum)

Same phenomena in (a) & (b) acc. to Einsteins eq. principle!

② Gravitational redshift seen as doppler shift.

Argumentation based on (a), acceleration.

Time for light from 1 to 2: $t = \frac{\Delta h}{c}$.

During this time t the speed of O_2 gained is $v = gt$.

Light from 1 w. freq. f is observed to be doppler shifted by O_2 to freq. f', and (see SR8): $f' = f\left(1 - \frac{v}{c}\right)$, if $\frac{v}{c} \ll 1$ (**)

Then: $\Delta f = f - f' = f\frac{v}{c} = f\frac{g\Delta h}{c^2}$ or $\frac{\Delta f}{f} = \frac{g\Delta h}{c^2}$

(**) Classical limit.

③ Gravitational redshift seen as energy loss of photon

Argumentation based on (b), gravitational field.

Close to surface of earth, planet or star gravitational potential energy is mgh, h height over surface.

Photon starts at 1 : Energy: $h\nu + mgx_1$

- and ends at 2 : Energy: $h\nu' + mgx_2$

Energy conservation: $h(\nu - \nu') = mg(x_2 - x_1)$

$\Delta\nu = \nu - \nu'$. $\Delta h = x_2 - x_1$ or $h\Delta\nu = mg\Delta h$ (*)

(* Care! h is Planck's constant. Δh is diff. in height!).

Mass of photon (SR0): $m = \frac{E}{c^2} = \frac{h\nu}{c^2}$

Then: $h\Delta\nu = \frac{h\nu}{c^2} g\Delta h$ or $\underbrace{\frac{\Delta\nu}{\nu} = \frac{g\Delta h}{c^2}}$.

If mass of planet M, radius R, at surface: $g = \frac{GM}{R^2}$.

(If far from surface, potential energy is $-G\frac{mM}{r}$
and then result would become $\frac{\Delta\nu}{\nu} = \frac{1}{c^2} GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$)

④ Gravitational time dilation. (**)

The redshift result in ③ seems somehow absurd.

If a source at 1 emits a given number of wavefronts per second (corresponding to frequency ν), how can a different number per second arrive at a receiver at 2 at rest with respect to the source?

This result only makes sense if source clock and receiver clock keep time at different rates.

(**) Formulations partly from: "Understanding Relativity" by Leo Sartori.

The source at 1 with frequency f emits N wavefronts during the timeinterval Δt as measured by a clock at 1.
We have: $N = f \cdot \Delta t$.
When the N wavefronts arrive at the receiver at 2, they arrive with a smaller frequency f' during the timeinterval $\Delta t'$ as measured by a clock at 2, and $N = f' \Delta t'$.
So: $f' \Delta t' = N = f \Delta t$, and $f' < f \Rightarrow \Delta t' > \Delta t$.
The observer at 2 and the source at 1 are at rest relative to each other, so all wavefronts travel the same distance at same speed (c).
The observer at 2 must conclude that the whole process at 1 (emitting N wavefronts) must have happened during the time $\Delta t'$; and as $\Delta t' > \Delta t$, then the observer at 2 must conclude that the clock at 1 keeps time at a slower rate than the clock at 2.

(5) Schwarzschild gravitational time dilation formula.

$$\Delta t = \Delta t_0 \frac{1}{\sqrt{1 - \frac{2GM}{c^2 r}}} = \Delta t_0 \frac{1}{\sqrt{1 - \frac{R_s}{r}}}$$

(Se Data Booklet)

Here: r distance to center of planet or star
M mass of planet or star
$R_s = \frac{2GM}{c^2}$, Schwarzschild radius $\quad r > R_s$
Δt_0 "proper time" if no gravitation
Δt "dilated time", in gravitation field.

If $r \to \infty$ (no gravitation), then $\Delta t \approx \Delta t_0$
If $r \to R_s$ then $\Delta t \to \infty$.

⑥ Evaluation of time dilatation, gravitational, approximation

From SR10: $\frac{\Delta f}{f} = \frac{1}{c^2} GM \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$.

If $r_2 \to \infty$, no gravitation = proper time. $\frac{\Delta f}{f} = \frac{1}{c^2} GM \frac{1}{r}$
(we call $r_1 = r$ now).

From SR11: $f' \Delta t' = N = f \Delta t$, Δt being the proper time.

Now: $\frac{\Delta f}{f} = \frac{f - f'}{f} = 1 - \frac{f'}{f} = \frac{1}{c^2} GM \frac{1}{r}$

or $\frac{f'}{f} = 1 - \frac{1}{c^2} \frac{GM}{r}$

Dilated time $= \Delta t' = \frac{f}{f'} \Delta t = \frac{1}{1 - \frac{GM}{c^2 r}} \Delta t$

This approximation can be shown to be valid only in weak gravitational fields for which $g = \frac{GM}{r^2}$ small.
Then also $x = \frac{GM}{c^2 r}$ small.

For small x: $\frac{1}{1-x} \approx 1 + x$, so $\underline{\Delta t' \approx \left(1 + \frac{GM}{c^2 r}\right) \Delta t}$

The exact solution (Schwarzschild) is: $\Delta t' = \left(1 - \frac{2GM}{c^2 r}\right)^{-\frac{1}{2}} \Delta t$
(SR11)

For small x: $(1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2} x$
so in weak gravitation fields we get:

$$\Delta t' = \left(1 - 2\frac{GM}{c^2 r}\right)^{-\frac{1}{2}} \Delta t \approx \left(1 + \frac{GM}{c^2 r}\right) \Delta t.$$